Team

Problem 1. Consider the mixed problem for $u(t,x) = u(t,x_1,x_2,x_3)$:

$$\begin{cases} \Box u = 0 & \text{for } t > 0, x_3 > 0, \\ u|_{t=0} = g(x), \quad u_t|_{t=0} = h(x) & \text{for } x_3 > 0, \\ Mu|_{x_3=0} = 0 & \text{for } t > 0, \end{cases} \tag{***}$$

where $\Box = \partial_t^2 - \Delta$, and M denotes a first-order transport operator of the form

$$M = \partial_t + \sum_{i=1}^{3} b_i \partial_{x_i} + c$$

with constant coefficients (b_1, b_2, b_3) and c, and g, h vanish for all sufficiently small $x_3 > 0$. Assume that u, g, h are smooth and bounded and that $b_3 \leq 0$.

(i) Prove that $w = Mu$ for $t > 0, x_3 > 0$ also satisfies the wave equation:

$$\Box w = 0,$$

and then determine $w(t,x)$ for $t > 0, x_3 > 0$ from its initial and boundary conditions as a solution of $\Box w = 0$.

(ii) Find the solution $u(t,x)$ of problem (***) through the solution of the following transport equation:

$$\begin{cases} Mu = w & \text{for } t > 0,\ x_3 > 0, \\ u|_{t=0} = g, \end{cases}$$

where $w(t,x)$ is your solution to part (i) above in this problem.

Problem 2. Assume that A and B are bounded self-adjoint linear operators on a Hilbert space. If $A \geq B \geq 0$, then there is a linear operator T such that

$$B = T^* A T.$$

Problem 3. Consider the upper half plane $\mathbb{H} = \{ z \in \mathbb{C} : \operatorname{Im}(z) > 0 \}$. Let $k > 2$ and let $f \colon \mathbb{H} \to \mathbb{C}$ be a holomorphic function satisfying

$$\int_{\mathbb{H}} y^{k-2} \, |f(x+iy)| \, \mathrm{d}x \, \mathrm{d}y < +\infty.$$

Prove that $y^k f(x+iy)$ is bounded on $\mathbb{H}$.